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THE INTERPLAY BETWEEN DESIGN AND MATHEMATICS: INTRODUCTION AND BOOTSTRAPPING EFFECTS

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ABSTRACT

In spite of common perceptions, design and mathematics have much in common in the way they are practiced and in their results. Understanding the interplay between design and mathematics could therefore, lead to mutual benefits to both disciplines. This paper introduces this subject and focuses on one benefit that could arise from a tight transfer of knowledge between design and mathematics that bootstraps progress in both disciplines.

INTRODUCTION

Mathematics is a fundamental tool in engineering. It is being taught in all engineering disciplines and used on a regular basis to model and analyze diverse phenomena and engineered products. It is sometimes mistakenly perceived as the most important aspect of engineering or as its core knowledge base. Yet, engineering is much more than analysis. In fact, its core activity is design: the creation of new products to satisfy some need.

If we try to relate design and mathematics, they seem two very different disciplines. Mathematics seems strict and formal with logic playing key role in proving new theorems. In contrast, design involves a great deal of creativity and qualitative judgment that seem to escape formality and logic or rationality, which are not viewed as the fundamental driving force in design.[1]

However, a close examination of concepts and the practice of mathematics and design reveals that their relationships are much more interesting and intricate than the service that mathematics provides to engineering analysis.

This paper reviews the relations between design and mathematics but focuses on a specific relation: the bootstrapping between these disciplines. We define bootstrapping as an effect that happens between two entities in which one improves the second, which in turn, could use that improvement to improve the first entity. We show that concepts developed in engineering in one problem domain (Assur groups in kinematics) could be transformed to mathematics. Once in mathematics, these concepts could be used to address difficult problems in rigidity theory; shed light on the existence of previously unknown classes in rigidity theory; and provide additional techniques to prove complicated theorems.[2] The proof process reveals ideas that in turn, could be used to advance ideas in another engineering problem domain – tensegrity structures. We anticipate that now, new insight would be developed in mathematics from subsequent bootstrapping iterations. By using infused design, all the insight generated in such bootstrapping scenarios could be transformed to related engineering domains**.**

RELATIONS BETWEEN MATHEMATICS AND DESIGN

The relations between mathematics and design could be described through two interacting perspectives: *process* and *product* (see Figure 1). The process perspective deals with the processes that professionals in both disciplines follow when they solve their professional problems, e.g., designing products or proving theorems. The product perspective deals with the

[1] Notwithstanding positions or studies aiming to advance mathematics or formality as a foundation of design.

[2] Rigidity theory has practical implications to biology and chemistry, among other disciplines.

outcome artifact of the process, be it a real product or a mathematical theorem.

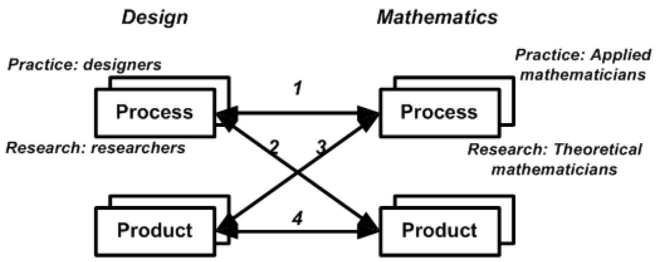

Figure 1: Relations between design and mathematics

Through the analysis, it is important to distinguish between people who advance these two disciplines (e.g., design researchers or mathematicians) and people who use the knowledge generated in these disciplines (e.g., designers or applied mathematicians). This distinction complicates the analysis because when design researchers develop a method in their research, this method is the *product* of their work, yet it becomes the basis for a *process* for designers. Altogether, the two perspectives and the type of people create 16 bidirectional interactions between the four design elements and the four mathematics elements.

While all these interactions are possible and in both directions, they might not be equally important; for example, we anticipate that interaction 1 will be more important from design to mathematics than in the opposite direction (meaning that design processes would teach us more about mathematical processes than vice versa). In addition, we do not foresee interaction 3 playing much role unless the design product is a design tool used to support mathematical reasoning processes.[3] Similarly, interaction 2 will probably not work directly from design process to mathematics product. The opposite direction of 2 is the traditional use of mathematical models to support engineering, e.g., topology concepts for supporting the modeling of design processes (Braha and Reich, 2003). Since we cannot cover all interactions in an introductory paper, we will provide several examples to motivate the further study of the relations between mathematics and design.

Figure 2 provides several examples that are discussed in this paper. The first four, (a), (b), (c) and (d) are briefly discussed and the fifth (e) is detailed in a separate section.

(a) Impact of design processes on mathematics processes. As we mentioned in the introduction, a prevalent perspective holds that mathematics is about proving theorems using logic or other formalisms (Long, 1986). Also pervasive in the general pubic is the view that design is creative potentially chaotic process, while customary view of design academics is that formal studies are fundamental to engineering design rather than creativity or qualitative judgment. Only lately do engineering academics begin to share the value of creativity as central to engineering design. Yet, it turns out that the gap between design and mathematics processes is not so wide.

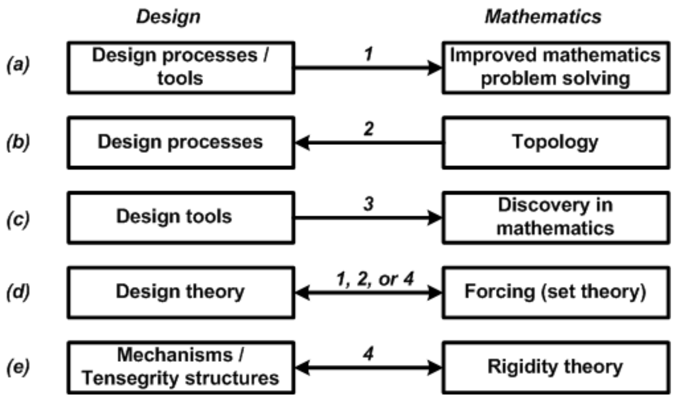

Figure 2: Examples of interactions between design and mathematics. The number on the arrow denotes the kind of interaction from Figure 1.

First, the process of developing mathematics is similar to developing or designing products. The practice of mathematics is rather social and not formal as perceived (Lakatos, 1976; De Millo *et al*., 1979). Farmer and Mohrenschildt (2003) describe mathematics as a process consisting of three steps: (1) *Model creation* for building mathematical models of the world; (2) *Model exploration* for exploring models by stating and proving conjectures and performing computations; and (3) *Model connection* to each other so that results obtained in one model could be used in other models. In this description, the act of formal proving theorems is quite minimal. These steps roughly correspond to design process steps if we use a slightly different terminology. We propose that design and (real) mathematics is a process with roughly 3 stages: (1) problem formulation and conceptual design (corresponding to model creation); (2) detailed design and production (corresponding to model exploration); and (3) integration or sales (corresponding to model connection). Therefore, design processes and strategies, and tools to manage them could be used to advance the practice of mathematics.

Even if we concentrated on the proving process, the story[4] of Fermat's last theorem can be used to demonstrate the design qualities of mathematical proofs: the process includes creating prototypes (proofs of special cases that cannot generalize); complete products that fail in the market (failed proof: Yoichi Miyaoka's proof[5]); an imperfect product (Wiles proof with a flaw); and an improved product (Wiles, 1995; Wiles and

[3] One exception might be the influence of recursion as a method in proving mathematical theorems or solving mathematical problems in designing effective algorithms in computer science or even in developing multi-level systems.

[4] Easy accessible story by David Shay appears in http://www.geocities.com/fermatnow/flt/index.htm.

[5] Science Magazine reported on this proof while it was checked before disproved as follows: "For the first time in memory, the mathematics community is optimistic that its most famous open problem - Fermat's Last Theorem - may finally have been proved. … Although no one will be completely confident until all the details have been thoroughly checked, those involved feel that Miaoka's proof has the best chance yet of settling the centuries-old problem."

Taylor, 1995). In executing these proof trials, diverse models and intermediate by-products are created and discarded. This clearly confirms that the proving process is anything but a clear formal directed derivation from the theorem statement.

There are many other examples of such practice in the history of mathematics, e.g., the work of Euclid and its subsequent partial refutation (Handal, 2003) or the work of Euler on infinite series (Kline, 1983). Ignoring the failed paths and the incorrect proofs in mathematics leads to a wrong perception of mathematics practice (Long, 1986).

Besides observing that the proving process also involves significant design, the value of such observation could be significant. For example, product development projects often create numerous by-products that could be used subsequently to improve profitability or advance other products. This has been demonstrated, for example, by the US Space project. While some may doubt the direct value of landing on the moon, the project clearly generated many diverse technologies that were subsequently used in remote areas. Similarly, if mathematics processes and their by-products are managed, they could be used to derive value in other disciplines or applications. Later in this paper, we demonstrate one such transfer.

(b) Use of mathematical concepts to model design processes. This example demonstrates the evolution of design process models utilizing different variations of a mathematical concept. The mathematical concept of topology was first used to model design processes by Yoshikawa (1981). It was the point-set topology concept used to model ideal processes. This model was extended to more general processes (Tomiyama and Yoshikawa, 1986), but was still limited (Tomiyama, 1994). A more general model using closure spaces was subsequently proposed to allow modeling real design processes (Braha and Reich, 2003).

(c) Use of design tools to support mathematical discovery. This example is rather specific but demonstrates that by considering the spectrum of interaction between design and mathematics, new practices could emerge in mathematics. For example, variable neighborhood search (Hansen and Mladenovic, 2003) is a method for solving complex optimization problem by local search. System employing this method could be used to generate empirically, data that could be analyzed and lead to conjectures in graph theory. Hansen and Mladenovic reviewed such examples in which the key issue was based in the design stage: the generation of interesting conjectures that were subsequently proven easily.

(d) Use of mathematical concepts to support design theory and conversely. Clearly, mathematics tools are used on a regular basis by designers in different design stages. For researchers developing design theories, mathematics formalisms are also indispensable. Discrete mathematical models play critical role in modeling engineering systems and processes. They are the basis of much work that appears later in this paper. But recent advances in design theory explore deep relationships with fundamental issues in mathematics.

New design theories like C-K theory (Hatchuel and Weil, 2003, 2007a) capture the creation of creative concepts by mobilizing non standard set theories (like ZF without AC).[6] These theories warrant the existence of classes of objects which cannot be described as classic sets and which are crucial for design thinking. For instance, designers want to consistently speak of the collection of "fuel cells for domestic use" when they envision to design some of them. Such collection is obviously not a classic set (most elements are undefined), yet Designers have to formulate propositions about such objects without generating nonsense.

Moreover, some fundamental topics in mathematics seem to be exactly a design theory of special mathematical objects. For example, Forcing in Set theory describes the generation of new models of standard set theory (collection of well formed sets) from existing ones. Forcing is used to prove independence theorems by generating sets that may or may not possess some property. Similarly, a designer may prove the independence between function and form for some class of objects by designing different forms that achieve the same required functionalities. Beyond such intuitive analogy, it has been shown that the operations of Forcing correspond narrowly to the design operations described by C-K theory (Hatchuel and Weil, 2007b).

Thus, there is growing evidence that deep interrelations can be found between design theory and basic areas of mathematics (and logic). On one hand, Forcing supports these new design theories. On the other hand, design theory highlights the universality of Forcing and its possible extension to the creative design of objects of the real world. This is a first example of bootstrapping between design and mathematics, which will be further discussed in the area of engineering systems.

(e) Bootstrapping design and mathematics knowledge. Research in design and mathematics could bootstrap each other if results from one discipline are used to advance the other discipline whose results, in turn, are transferred back, providing new ways to advance or solve tough unsolved problems. This example demonstrates that by-products of design or mathematics processes could be valuable if observed well and if mechanisms to transfer them to diverse disciplines exist.

EXAMPLE: BOOTSTRAPPING BETWEEN MATHEMATICS AND ENGINEERING DESIGN

Infused design (ID) is a method for supporting collaborative design of professionals from diverse disciplines on multi-disciplinary products (Shai and Reich, 2004a,b; Shai *et al*., 2007); it enhances communication between disciplines and could lead to creative design of new products which, in turn, could foster innovation in other disciplines.

Research in this area has resulted new design methods as well as products (e.g., mechanical transistor and rectifiers).

[6] The letters C-K stand for Concept and Knowledge respectively; ZF is a shortcut for Zermelo-Fraenkel set theory; and AC is the Axiom of Choice.

One application in which ID is being presently applied is developing a method for finding all the topologies of 2D tensegrity structures. The problem is to characterize and find a method that will enable engineers to construct all the different topologies of tensegrity structures. This capability is also central to synthesizing their geometry based on given topology and other constraints including *a priori* determination of their stability. These are known to be difficult tasks without existing general solutions. This work is very important since this type of structures have diverse practical applications, including Deployable Tensegrity Structures shown in Figure 3.

(a)

(b)

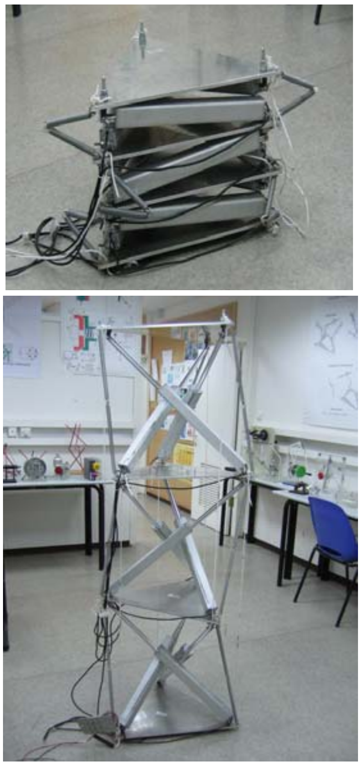

Figure 3: A deployable tensegrity structure: (a) contracted structure, (b) deployed structure

The process of revealing this method involved interplay between an engineering group (EG), whose activities were mentioned before, and a mathematician group (MG) that works on rigidity theory. Rigidity theory is a branch of mathematics that originated by classical studies on rigid frameworks (Laman, 1970) but evolved into a study of abstract concepts of rigidity (e.g., in matroids) that might subsequently be applied to study rigidity in different applications but could also be advanced on their own within mathematics (Graver *et al.*, 1993).

The following example describes a bootstrapping scenario in which concepts and knowledge transferred between design and mathematics, leading to mutual fertilization and faster growth of knowledge in both disciplines. Figure 4 shows the flow of events in this example as they unfolded.

The EG worked on developing general methods for creating engineering systems. During this work, they were exposed to an old work developed in Russia by Assur in 1914. Assur proposed a method for decomposing each mechanism into basic components, termed Assur groups. It was also proved that there is a unique decomposition for each mechanism (1).

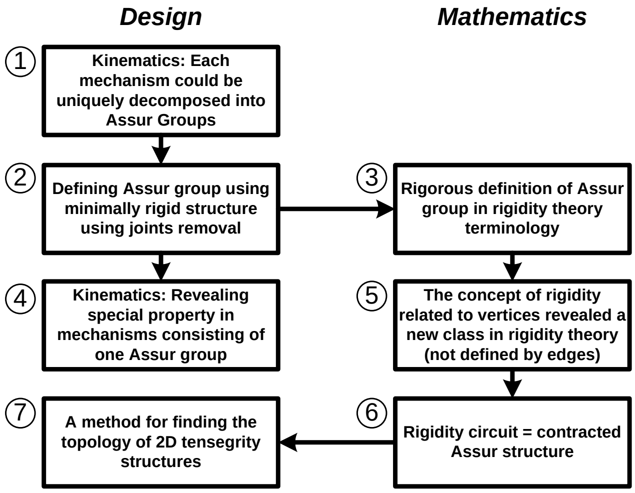

Figure 4: bootstrapping knowledge growth in design and mathematics

The first step that enabled the collaboration work with the mathematicians was when a new definition of Assur groups was revealed, this time using basic terms from graph and rigidity theories (2). This bootstrapping step led to progress in the two groups. In mathematics it enabled defining all the Assur groups in terms of graph and rigidity theories (3), thus augmenting to this topic mathematics knowledge developed for other topics.

While working on this systems approach, the EG observed that each mechanism consisting of exactly one Assur group possesses a property that distinguished it from mechanisms consisting of several Assur groups (4). It became apparent that it was very difficult to prove this property using only known engineering methods; however, this property was proven through the interplay with the MG, who used existing theorems that were developed by mathematicians for other purposes.

It was observed that the definition of Assur groups has a special property in rigidity theory – minimal rigidity related to vertices. In the MG, they widely use the term 'minimum rigid related to edges (bars)', i.e., the structure is rigid if deletion of any edge (bar) infects its rigidity (5). This new definition enables viewing rigid graphs differently.

Now, at the mathematics level it was revealed that contracted Assur structures obtained after replacing all the pin joints with one joint, result in a rigidity circuit for which there exist known theorems in rigidity theory that says that there is a unique self-stress situation in all the rods (6). This paves a new way to find all the topologies of tensegrity structures in 2D (7).

While the precise details of the aforementioned steps are beyond the scope of this paper, it is clear that the interaction between mathematics and design led to new knowledge in both disciplines. Practitioners in both disciplines struggled to find solutions to problems that were answered through the bootstrapping process. By using ID, those answers could further translate to other disciplines.

DISCUSSION AND SUMMARY

Studying the interplay between mathematics and design requires also appreciating the differences between these disciplines and their meanings. For example, the statement of the end results of these disciplines, theorems or products, whether perfect or imperfect. With reference to Wiles 1st attempt to prove Fermat's last theorem, André Weil (1994) stated: "I believe he has had some good ideas in trying to construct the proof but the proof is not there. To some extent, proving Fermat's Theorem is like climbing Everest. If a man wants to climb Everest and falls short of it by 100 yards, he has not climbed Everest."

This statement points to a major difference between design and mathematics. Design does not concern itself with flawless products. In fact, we are accustomed to use imperfect products all the time and get frustrated when they do not work. In contrast, mathematics is all-or-none discipline. Either you have the proof or you don't. This distinction is related to the *verification* of the product but not its validity. Well proven mathematical theorems could be as useless as working products.

Clearly, the bootstrapping process (Figure 4) is a collaborative effort made possible by the availability of a shared representation – the same representation that supports ID. The details of the exchange reveal the kind of choices, reasoning, failure paths, and processes that appear in design. Consequently, this example reaffirms the fact that doing mathematics is closely related to design and much more than theorem proving.

Design plays key role in mathematics discourse (as well as in many other disciplines). It is central to human problem solving. If this is the case, we should give design its status and teach these disciplines with design practice in mind.

In summary, studying the interplay between mathematics and design promises to lead to advances in both and other related disciplines. This study could lead to viewing design in a broader, more fundamental manner than it presently enjoys.

Finally, the relations between design and mathematics discussed herein appears in similar ways when we analyze the relations between theory and practice in engineering design or our attempt at a meta-level to characterize how progress is made in science and engineering.

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